Computing the Lambert W function

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The Lambert W function is one of the immense zoology of special functions in mathematics. Formally, it is defined implicitly as an inverse function: If $w = W(x)$ is the Lambert W function, then we have

$$x = we^w \tag{1}$$

In the most general circumstances, W is a complex-valued function of a complex variable z. To keep things simple, we will look only at the real-variable case.

Fundamentally, computing W is a root finding problem: Given x, we can find $w = W(x)$ by finding the root (if one exists) of the function

$$F(w) = x - we^w$$

(The notation here is a little different than we are used to: x is a fixed parameter, and w is the variable we are trying to find to make F zero.) We can learn quite a bit about the W function by plotting F, which can be done by specifying w and then computing x from (1). The MATLAB commands for this might be:

```
w = [-400:200]*0.01;
x = w.*exp(w);
plot(x,w,'b-')
axis([-1 6 -4 2])
grid on
```

The last command lays a grid of dotted lines over the plot. The results are given in Fig. 1 (next page). The problem is that we can't easily compute $W(x)$ for an arbitrary x using this scheme, which is what we need to do.

We can deduce quite a bit about $W(x)$ from this graph, although some analysis would need to be done to confirm our deductions:

1. It appears that $W(x) \to \infty$ as $x \to \infty$;
2. It looks like $W(0) = 0$ (and that is easy to confirm from (1);
3. The derivative $W'(x)$ appears to be infinite for x a little more than $-1/2$, and W appears to be -1 at this point (a little work with calculus can establish that $W'(x) \to \infty$ as $x \to -e^{-1}$, and $W(-e^{-1}) = -1$);

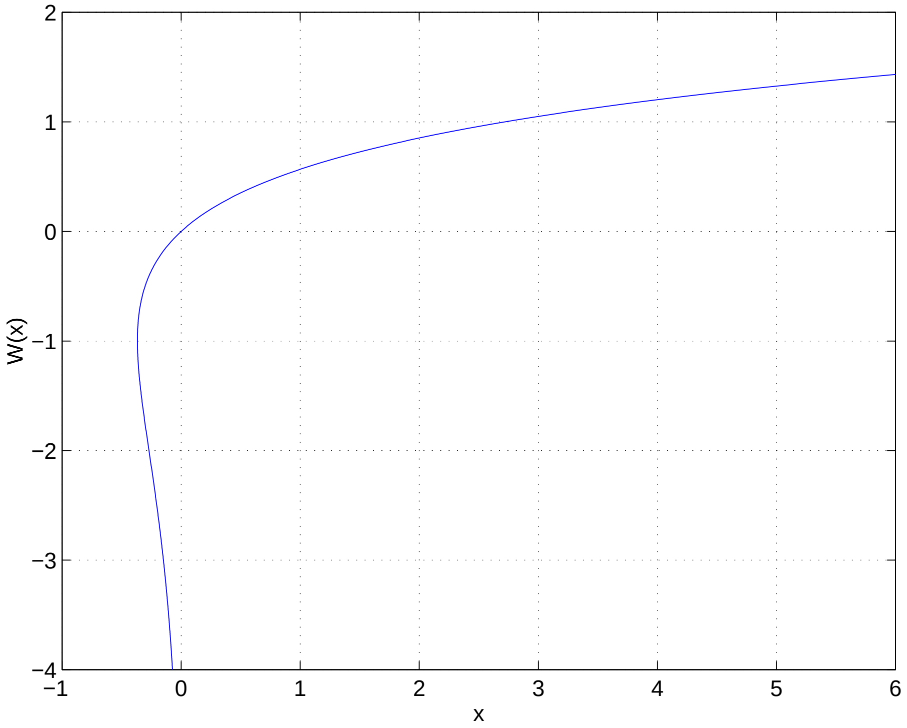

Figure 1: First plot of Lambert W function

4. W looks to be double-valued and asymptotic to $-\infty$ for $x < 0$.

To compute W values, we used Newton's method. For $x > 0$, we computed out to $x = 20$ in increments of 0.1, using $W(x_{k-1})$ as the initial value for the computation of $W(x_k)$. For $-e^{-1} < x < 0$, we did essentially the same thing (with a different increment), except we marched from $x = -e^{-1}$ to $x = 0$, again using the result of the previous iteration as the starting value for the next iteration, and similarly for $x < -e^{-1}$. The Newton iteration in each case is (remember $F' = \frac{dF}{dw}$)

$$\begin{aligned} w_{n+1} &= w_n - \frac{F(w_n)}{F'(w_n)}, \\ &= w_n - \left(\frac{x - w_n e^{w_n}}{-e^{w_n} - w_n e^{w_n}} \right), \\ &= w_n + \left(\frac{x e^{-w_n} - w_n}{1 + w_n} \right), \end{aligned}$$

and a little algebra simplifies this to:

$$w_{n+1} = w_n^2 + \frac{x e^{-w_n}}{1 + w_n}.$$

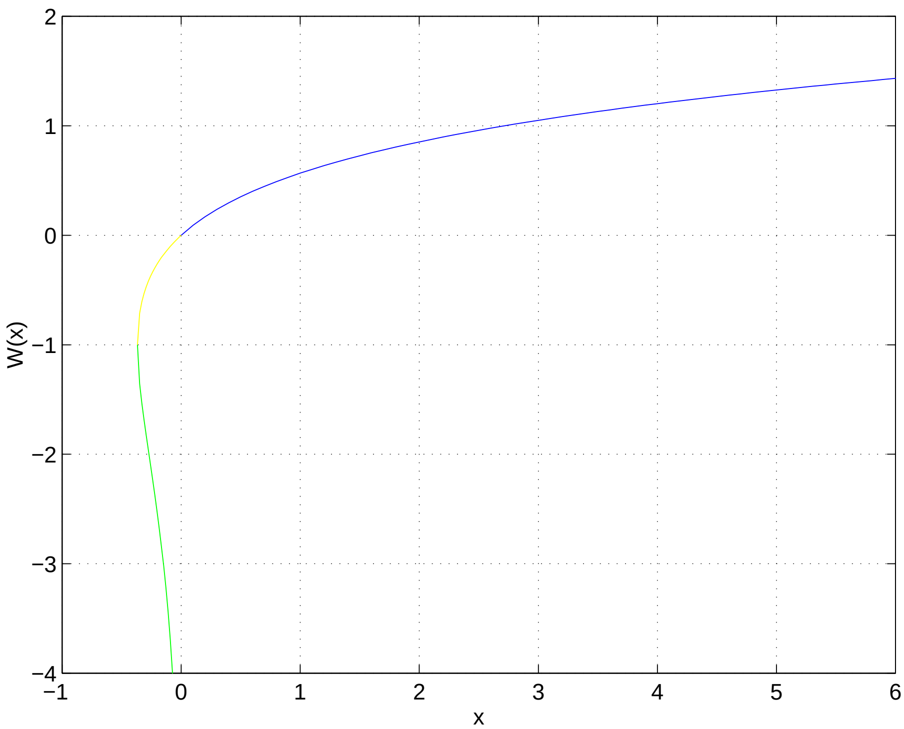

Figure 2: Plot of Lambert W function as computed by Newton's method

The initial guess for each point was the converged value of the previous point, using the two known values $W(0) = 0$ and $W(-e^{-1}) = -1$, and the iteration converged (to 10 digits) in at most 6 iterations.

The graph of $W(x)$ is given in Fig. 2, with each segment of the domain colored differently.

Unfortunately, this program is not what we need if all we want to do is compute a single value of $W(x)$. Let's now construct a MATLAB m-file that returns the value of $W(x)$ when given x and a branch indicator. (Remember, W is multiple-valued for $-\infty < x < 0$, so we need to allow for that.) The only issue is to construct a means to compute good initial values for the Newton iteration. For $x > -1/e$ (the blue and yellow portions of the curve), the graph looks a lot like a translated square root function. For $x < -1/e$ (the green branch) the graph looks a lot like a translated reciprocal function. We will use these to build our "initial guess generators."

Consider the function $f(x) = -1 + \sqrt{x + e^{-1}}$, defined for $x \geq -e^{-1}$, as well as $g(x) = e^{-1}/x$, defined for $x \leq -e^{-1}$. These are plotted as red lines in Fig. 3. While not wonderfully close to the graph of the Lambert W function, they look to be close enough to obtain our initial guesses.

To complete our work here we wrote a MATLAB m-file to compute $W(x)$ using

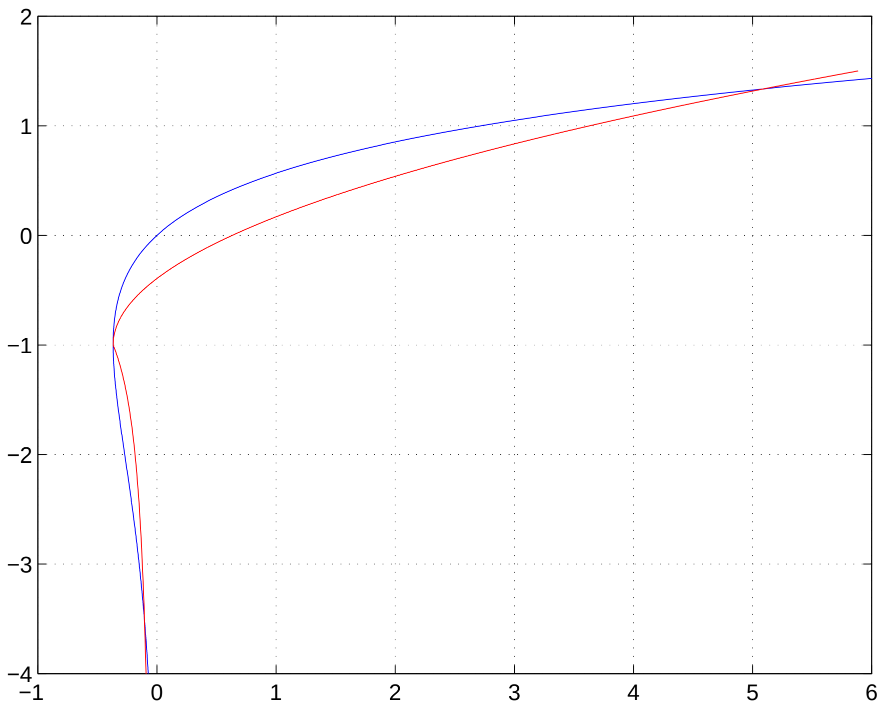

Figure 3: Lambert W function (b) with initial guess generators (r)

f and g (given above) to generate the initial guesses for Newton's method. To accommodate the fact that W is multiple-valued, we wrote the code to return two values (assuming the user would choose the proper one for their application). If $x > 0$, the second returned value is set to 0. For evaluation purposes, only, we also returned the number of iterations to convergence; the maximum number of iterations observed for $-e^{-1} < x < 290$ was 6.